RELATIVE SIZE OF SUBSETS OF A SEMIGROUP

IGOR PROTASOV AND SERHII SLOBODIANIUK

Abstract. Given a semigroup S, we introduce relative (with respect to a filter τ on S) versions of large, thick and prethick subsets of S, give the ultrafilter characterizations of these subsets and explain how large could be some cell in a finite partition of a subset $A \in \tau$.

1. Introduction

For a semigroup S, $a \in S$, $A \subseteq S$ and $B \subseteq S$, we use the standard notations

$$a^{-1}B = \{x \in S : ax \in B\}, \quad A^{-1}B = \bigcup_{a \in A} a^{-1}B,$$

$$[A]^{<\omega} = \{F \subseteq A, F \text{ is finite}\}.$$

A subset A of S is called

- *large* if there exists $F \in [S]^{<\omega}$ such that $S = F^{-1}A$;
- *thick* if, for every $F \in [S]^{<\omega}$, there exists $x \in S$ such that $Fx \subseteq A$;
- *prethick* if $F^{-1}A$ is thick for some $F \in [S]^{<\omega}$;
- *small* if $L \setminus A$ is large for any large subset L.

In the dynamical terminology [8, p.101], large and prethick subsets are known as syndedic and piecewise syndedic. These and several other combinatorially rich subsets of a semigroup are intensively studied in connection with the Ramsey Theory (see [8, Part III]). In [6], large, thick and prethick subsets are called right syndedic, right thick and right piecewise syndedic.

The names large and small subsets of a group appeared in [4], [5] with additional adjective "left". Unexplicitely, thick subsets were used in [11] to partition an infinite totally bounded topological group G into $|G|$ dense subsets. For more delicate classification of subsets of a group by their size, we address the reader to [3], [9], [10], [14], [17], [18].

In frames of general asymptology [20, Chapter 9], large and thick subsets of a group could be considered as counterparts of dense and open subsets of a topological space.

Our initial motivation to this note was a desire to refine and generalize to semigroup the following statement [13, Corollary 3.4]: if a neighborhood U of the identity e of a topological group G is finitely partitioned then there exists a cell A of the partition and a finite subset $F \subset U$ such that FAA^{-1} is a neighborhood of e. On this way, we run to some relative (with respect to a filter) versions of above definitions.

Let S be a semigroup and let τ be a filter on S. We say that a subset A of S is

- τ-*large* if, for every $U \in \tau$, there exists $F \subseteq [U]^{<\omega}$ such that $F^{-1}A \in \tau$;
- τ-*thick* if there exists $U \in \tau$ such that, for any $F \in [U]^{<w}$ and $V \in \tau$, one can find $x \in V$ such that $Fx \subseteq A$;
- τ-*prethick* if, for every $U \in \tau$, there exists $F \in [U]^{<\omega}$ such that $F^{-1}A$ is τ-prethick;
- τ-*small* if $L \setminus A$ is τ-large for every τ-large subset L.

In the case $\tau = \{S\}$, we omit τ and get the seminal classification of subsets of S by their size.

To conclude the introduction, we need some algebra in the Stone-Čech compactifications from [8].

2010 *Mathematics Subject Classification.* 05D10, 20M12, 22A15.

Key words and phrases. large, thick and prethick subsets of a semigroup; ultrafilters; the Stone-Čech compactification of a discrete semigroups; minimal left ideal.

For a discrete semigroup S. we consider the Stone-Čech compactification βS of S as the set of all ultrafilters on S, identifying S with the set of all principal ultrafilters, and denote $S^* = \beta S \setminus S$. For a subset A of S and a filter τ on S, we set

$$\overline{A} = \{p \in \beta S : A \in p\}, \quad \overline{\tau} = \bigcap \{\overline{A} : A \in \tau\} = \{p \in \beta S : \tau \subseteq p\}$$

and note that the family $\{\overline{A} : A \subseteq S\}$ forms base for the open sets on βS, and each non-empty closed subset in βS is of the form $\overline{\tau}$ for an appropriate filter τ on S.

The universal property of the Stone-Čech compactifications of discrete spaces allows to extend multiplication from S to βS in such way that, for any $p \in \beta S$ and $g \in S$ the shifts $x \mapsto xp$ and $x \mapsto gx$, $x \in \beta S$ are continuous.

For any $A \subseteq S$ and $q \in \beta S$, we denote

$$A_q = \{x \in S : x^{-1}A \in q\}.$$

Then formally the product pq of ultrafilters p and q can be defined [8, p.89] by the rule:

$$A \in pq \leftrightarrow A_q \in p.$$

In this note, we give the ultrafilter characterizations of τ-large and τ-thick subsets (section 2) and τ-prethick subsets (section 3) in spirit of [6], [8], [18]. If τ is a subsemigroup of βS, we describe the minimal left ideal of $\overline{\tau}$ to understand how big could be the cells in a finite partition of a subset $A \in \tau$.

2. Relatively large and thick subsets

Let τ be a filter on a semigroup S.

Theorem 2.1. *A subset L of S is τ-large if and only if, for every $p \in \overline{\tau}$ and $U \in \tau$, we have $L_p \cap U \neq \varnothing$.*

Proof. We suppose that L is τ-large and take arbitrary $p \in \overline{\tau}$ and $U \in \tau$. We choose $F \in [U]^{<\omega}$ such that $F^{-1}L \in \tau$. Since $F^{-1}L = \bigcup_{g \in F} g^{-1}L$, there exists $g \in F$ such that $g^{-1}F \in p$ so $g \in L_p$ and $L_p \cap U \neq \varnothing$.

To prove the converse statement, we assume that L is not τ-large and choose $U \in \tau$ such that $F^{-1}L \notin \tau$ for every $F \in [U]^{<w}$. Then we take an ultrafilter $p \in \overline{\tau}$ such that $G \setminus F^{-1}L \in p$ for each $F \in [U]^{<\omega}$. Clearly, $g^{-1}L \notin p$ for every $g \in U$ so $U \cap L_p = \varnothing$. □

Theorem 2.2. *A subset T of S is τ-thick if and only if there exists $p \in \overline{\tau}$ such that $T_p \in \tau$.*

Proof. We suppose that T is τ-thick and pick corresponding $U \in \tau$. The set $[U]^{<w} \times \tau$ is directed $\leq$ by the rule:

$$(F, V) \leq (F', V') \leftrightarrow F \subseteq F', V' \subseteq V.$$

For each pair (F, V) we choose $g(F, V) \in V$ such that $Fg(F, V) \subseteq T$. The family

$$P_{F,V} = \{g(F', V') : (F, V) \leq (F', V')\}, \quad (F, V) \in [U]^{<\omega} \times \tau$$

is contained in some ultrafilter $p \in \overline{\tau}$. By the construction, $U \subseteq T_p$ so $T_p \in \tau$.

To prove the converse statement, we choose $p \in \overline{\tau}$ such that $T_p \in \tau$. Given any $F \in [T_p]^{<\omega}$ and $V \in \tau$, we take $P \in p$ such that $P \subseteq V$ and $gP \subseteq T$ for each $g \in F$. Then we choose an arbitrary $x \in P$ and get $Fx \subseteq T$ so T is τ-thick. □

We say that a subset T of S is τ-*extrathick* if $T_p \in \tau$ for each $p \in \overline{\tau}$.

By [6, Theorem 2.4], a subset T is thick if and only if T intersects each large subset non-trivially. In the case $\tau = \{G\}$, this is a partial case of the following theorem.

Theorem 2.3. *If each subset $U \in \tau$ is τ-extrathick then a subset T of S is τ-thick if and only if $T \cap L \cap U \neq \varnothing$ for any τ-large subset L and $U \in \tau$.*

Proof. We assume that T is τ-thick and use Theorem 2.2 to find $p \in \overline{\tau}$ such that $T_p \in \tau$. We take an arbitrary τ-large subset L and $U \in \tau$. Since U is τ-extrathick, we have $U_p \in \tau$. By Theorem 2.1, $L_p \cap (T_p \cap U_p) \neq \varnothing$. If $g \in L_p \cap T_p \cap U_p$ then $L \in gp$, $T \in gp$, $U \in gp$. Hence, $T \cap L \cap U \neq \varnothing$.

We suppose that $T \cap L \cap U = \varnothing$ for some τ-large subset L and $U \in \tau$ but T is τ-thick. We take $p \in \overline{\tau}$ such that $T_p \in \tau$. Since U is τ-extrathick, we have $U_p \in \tau$. By Theorem 2.1, $L_p \cap (T_p \cap U_p) \neq \varnothing$. If $g \in L_p \cap T_p \cap U_p$ then $L \in gp$, $T \in gp$, $U \in gp$. Hence, $T \cap L \cap U \neq \varnothing$ and we get a contradiction. □

Theorem 2.4. *Let $g \in S$ and let τ be a filter on S such that $g^{-1}U \in \tau$ for each $U \in \tau$. If a subset L of S is τ-large and a subset T of S is τ-thick then gL and $g^{-1}T$ are τ-large and τ-thick respectively.*

Proof. To prove that gL is τ-large, we take an arbitrary $U \in \tau$ and choose $V \in \tau$ such that $gV \subseteq U$ (using $g^{-1}U \in \tau$). Since L is τ-large, there is $F \in [V]^{<\omega}$ such that $F^{-1}L \in \tau$. We note that $F^{-1}L = (gF)^{-1}gF$. Since $gF \in [U]^{<\omega}$, we conclude that gF is τ-large.

To see that $g^{-1}T$ is τ-thick, we pick $U \in \tau$ such that, for every $F \in [U]^{<\omega}$ and $W \in \tau$, there is $x \in W$ such that $Fx \subseteq T$. We choose $V \in \tau$ such that $gV \subseteq U$. Then we take an arbitrary $H \in [V]^{<w}$ and $W \in \tau$. Since $gH \in [U]^{<\omega}$, there exists $y \in W$ such that $gHy \subseteq T$ so $Hy \subseteq g^{-1}T$ and $g^{-1}T$ is τ-thick. □

We say that a family $\mathcal{F}$ of subsets of S is *left (left inverse) invariant* if, for any $A \in \mathcal{F}$ and $g \in S$, we have $gA \in \mathcal{F}$ ($g^{-1}A \in \mathcal{F}$).

Corollary 2.1. *If τ is inverse invariant then the family of all τ-large (τ-thick) subsets is left (left inverse) invariant.*

Theorem 2.5. *Let τ be a filter on S such that, for every $U \in \tau$, we have $\{g \in S : g^{-1}U \in \tau\} \in \tau$. If T is τ-thick then there exists $V \in \tau$ such that $g^{-1}T$ is τ-thick for every $g \in V$.*

Proof. We take $U \in \tau$ such that, for any $K \in [U]^{<\omega}$ and $W \in \tau$, we have $Kx \subseteq T$ for some $x \in W$. Then we choose $V \in \tau$ such that, for every $g \in V$, there exists $V_g \in \tau$ with $gV_g \subseteq U$. Given any $F \in [V_g]^{<\omega}$ and $W \in \tau$, we pick $x \in W$ such that $gFx \subset T$ so $Fx \subseteq g^{-1}T$ and $g^{-1}T$ is τ-thick. □

A topology $\mathcal{T}$ on a semigroup S is called *left invariant* if each left shift $x \mapsto gx$, $g \in G$ is continuous (equivalently, the family $\mathcal{T}$ is left inverse invariant).

We assume that S has identity e and say that a filter τ on S is *left topological* if τ is the filter of neighborhoods of e for some (unique in the case if S is a group) left invariant topology $\mathcal{T}$ on S.

Let τ be a left topological filter on S. Then each subset $U \in \tau$ is τ-extrathick and τ satisfies Theorem 2.5. Hence, Theorems 2.3 and 2.5 hold for τ.

We show that Theorem 2.5 needs not to be true with τ-large subsets in place of τ-thick subsets even if τ is a filter on neighborhoods of the identity for some topological group.

We endow $\mathbb{R}$ with the natural topology, denote $\mathbb{R}^+ = \{r \in \mathbb{R}, r > 0\}$ and take the filter τ of neighborhoods of 0. The set $\mathbb{R}^+$ is τ-large because $\mathbb{R}^+ - x \in \tau$ for each $x \in \mathbb{R}^+$. On the other hand, $\mathbb{R}^+ + x$ is not τ-large for each $x \in \mathbb{R}^+$.

3. Relatively prethick subsets

We say that a filter τ on S is a *semigroup filter* if $\overline{\tau}$ is a subsemigroup of the semigroup βS and note that, if either τ is inverse left invariant or S has the identity and τ is left topological then τ is a semigroup filter.

In the case $\tau = \{S\}$, the following statement is Theorem 4.39 from [8].

Theorem 3.1. *Let τ be a semigroup filter on S. An ultrafilter $p \in \overline{\tau}$ belongs to some minimal left ideal L of $\overline{\tau}$ if and only if, for each $A \in p$, the set A_p is τ-large.*

Proof. Let L be a minimal left ideal of $\overline{\tau}$, $p \in L$, $A \in p$ and $U \in \tau$. Clearly, $L = \overline{\tau}p$. We take an arbitrary $r \in \tau$. By the minimality of L, $\overline{\tau}rp = \overline{\tau}p$, so there exists $q_r \in \tau$ such that $q_r rp = p$. Since $A \in q_r rp$ and $U \in q_r$, by the definition of the multiplication in βS, there exists $B_r \in r$ such that $\overline{B_r}p \subseteq \overline{x_r^{-1}A}$. We consider the open cover $\{\overline{B_r}, r \in \overline{\tau}\}$ of the compact space $\overline{\tau}$ and choose its finite

subcover $\{\overline{B}_r : r \in K\}$. We put $B = \bigcup_{r\in K} B_r$, $F = \{x_r, r \in K\}$. Then $B \in \tau$ and $B \subseteq (F^{-1}A)_p$. By the choice, $F \subseteq U$. Since p is an ultrafilter, we have $(F^{-1}A)_p = F^{-1}A_p$. Hence, A_p is τ-large.

To prove the converse statement we suppose that $\overline{\tau}p$ is not minimal and choose $r \in \overline{\tau}$ such that $p \notin \overline{\tau}rp$. Since the subset τrp is closed in $\overline{\tau}$, there exists $A \in p$ with $\overline{A} \cap \overline{\tau}rp = \varnothing$. It follows that $A \notin qrp$ for every $q \in \overline{\tau}$. Hence, $S \setminus A \in qrp$ for every $q \in \overline{\tau}$. It follows that there exists $U \in \tau$ such that $x^{-1}(G \setminus A) \in rp$ for each $x \in U$. By the assumption, there exists $F \in [U]^{<\omega}$ such that $F^{-1}A \in qp$ for every $q \in \overline{\tau}$. In particular, $x^{-1}A \in rp$ for some $x \in F$ and we get a contradiction. □

Corollary 3.1. *Let τ be a semigroup filter on S and let $p \in \overline{\tau}$ belongs to some minimal left ideal of $\overline{\tau}$. Then every subset $A \in p$ is τ-prethick.*

Proof. Given an arbitrary $U \in \tau$, we use Theorem 3.1 to find $F \in [V]^{<\omega}$ such that $(F^{-1}A)_p \in \tau$. By Theorem 2.2, $F^{-1}A$ is τ-thick. Hence, A is τ-prethick. □

Corollary 3.2. *Let τ be a semigroup filter on a group G and let $U \in \tau$. Then, for every finite partition $\mathcal{P}$ of U and every $V \in \tau$, there exists $A \in \mathcal{P}$ and $F \in [V]^{<\omega}$ such that $F^{-1}AA^{-1} \in \tau$.*

Proof. We take p from some minimal left ideal of $\overline{\tau}$. Then we choose $A \in \mathcal{P}$ such that $A \in p$. Applying Theorem 3.1, we find $F \in [V]^{<\omega}$ such that $(F^{-1}A)_p \in \tau$. If $x \in (F^{-1}A)_p$ then $F^{-1}A \in xp$ and $x \in F^{-1}AA^{-1}$. Hence, $F^{-1}AA^{-1} \in \tau$. □

In connection with Corollary 3.2, we would like to mention one of the most intriguing open problem in the subset combinatorics of groups [12, Problem 13.44] possed by the first author in 1995: given any group G, $n \in \mathbb{N}$ and partition $\mathcal{P}$ on G into n cells, do there exit $A \in \mathcal{P}$ and $F \subseteq G$ such that $G = FAA^{-1}$ and $|F| \leq n$? For recent state of this problem see the survey [2].

On the other hand [1], if an infinite group G is either amenable or countable, then for every $n \in \mathbb{N}$, there exists a partition $G = A \cup B$ such that FA and FB are not thick for each F with $|F| \leq n$. We do not know whether such a 2-partition exists for any uncountable group G and $n \in \mathbb{N}$.

Theorem 3.2. *Let G be a group, τ be a filter of neighborhoods of the identity for some group topology on G and $U \in \tau$. Then, for any partition $\mathcal{P}$ of U, $|\mathcal{P}| = n$ and $V \in \tau$, there exist $A \in \mathcal{P}$ and $K \subseteq V$ such that $KAA^{-1} \in \tau$ and $K \leq 2^{2^{n-1}-1}$.*

Proof. We consider only the case $n = 2$. For $n > 2$, the reader can adopt the inductive arguments from [16, pp. 120-121], where this fact was proved for $\tau = \{G\}$. So let $U = A \cup B$ and $e \in B$. We choose $W \in \tau$ such that $WW \subseteq U$ and denote $C = A \cap W$. If there exists $H \in \tau$ such that $xC \cap C \neq \varnothing$ for each $x \in H$ then $CC^{-1} \in \tau$ and we put $F = \{e\}$, so $F^{-1}AA^{-1} \in \tau$. Otherwise, we take $g \in V \cap W$ such that $gC \cap C = \varnothing$. Then $gC \subseteq WW \subseteq U$, so $gC \subseteq B$ and $B \cup g^{-1}B \in \tau$. We put $F = \{e, g\}$. Since $e \in B$, we have $F^{-1}BB^{-1} \in \tau$. □

Recall that a family $\mathcal{F}$ of subsets of a set X is *partition regular* if, for every $A \in \mathcal{F}$ and any finite partition of A, at least one cell of the partition is a member of $\mathcal{F}$.

For a subsemigroup filter $\overline{\tau}$ on S, we denote by $M(\overline{\tau})$ the union of all minimal left ideals of $\overline{\tau}$. In the case $\tau = \{G\}$, the following statement is Theorem 4.40 from [8].

Theorem 3.3. *Let τ be left inverse invariant filter on a semigroup S. Then the following statements hold*

(i) *a subset A of S is τ-prethick if and only if $\overline{A} \cap M(\overline{\tau}) \neq \varnothing$;*
(ii) *$P \in clM(\overline{\tau})$ if and only if each $A \in p$ is τ-prethick;*
(iii) *the family of all τ-prethick subsets of S is partition regular.*

Proof. (*i*) If $\overline{A} \cap M(\overline{\tau}) \neq \varnothing$ then A is τ-prethick by Corollary 3.1.

Assume that A is τ-prethick and pick a finite subset F such that $F^{-1}A$ is τ-thick. We use Theorem 2.2 to find $p \in \overline{\tau}$ such that $(F^{-1}A)_p \in \tau$. Then $F^{-1}A \in qp$ for every $q \in \overline{\tau}$. The set $\overline{\tau}p$ contains some minimal left ideal L of $\overline{\tau}$. We take any $r \in L$ so $F^{-1}A \in r$ and $A \in tr$ for some $t \in F$. Since τ is inverse left invariant $tr \in \overline{\tau}$. Hence, $tr \in M(\overline{\tau}) \cap \overline{A}$.

The statements (*ii*) and (*iii*) follow directly from (*i*). □

Theorem 3.4. *Let τ be a left invariant filter on a group G. A subset A of G is τ-prethick if and only if A is not τ-small.*

Proof. By the definition and Theorem 2.4, the family of all τ-small subsets of G is left invariant and invariant under finite unions. We suppose that A is τ-small and τ-prethick and take $K \in [G]^{<w}$ such that KA is τ-thick. We note that G is τ-large and KA is τ-small so $G \setminus KA$ is τ-large. But $(G \setminus KA) \cap KA = \varnothing$ and we get a contradiction with Theorem 2.3. □

We do not know whether Theorems 3.3 and 3.4 hold for any left topological filter τ (even for filters of neighborhoods of identity of topological groups).

For a subset A of an infinite group G, we denote

$$\Delta(A) = \{x \in G : xA \cap A \text{ is infinite}\}.$$

Answering a question from [15], Erde proved [7] that if A is prethick then $\Delta(A)$ is large. We conclude the paper with some relative version of this statement.

For a filter τ on a semigroup S and $A \subseteq S$, we denote

$$\Delta_\tau(A) = \{x \in S : (x^{-1}A \cap A) \cap U \neq \varnothing \text{ for any } U \in \tau\}.$$

In the case of a group G, $\Delta(A) = (\Delta(A))^{-1}$ so we have $\Delta(A) = \Delta_\tau(A)$ for the filter τ of all cofinite subsets of G.

Theorem 3.5. *Let τ be a left inverse invariant filter on a semigroup S. If a subset A of S is τ-prethick then $\Delta_\tau(A)$ is τ-large.*

Proof. We observe that $\Delta_\tau(A) = \bigcup\{A_p : p \in \overline{\tau}, A \in p\}$. Now let A be τ-prethick. We use Theorem 3.3(i) to find $p \in \overline{A} \cap M(\overline{\tau})$. By Theorem 3.1, for every $U \in \tau$, there exists a finite subset $K \subseteq U$ such that $K^{-1}A_p \in \tau$. Since $A_p \subseteq \Delta_\tau(A)$, we have $K^{-1}\Delta_\tau(A) \in \tau$, so $\Delta_\tau(A)$ is τ-large. □

Let τ be a left invariant filter on a group G and let $X \subseteq G$. Then $\Delta_\tau(X) = \{g \in G : (gX \cap X) \cap U \neq \varnothing \text{ for each } U \in \tau\}$ and $\Delta_\tau(X \setminus U) = \Delta_\tau(X)$ for each $U \in \tau$. Now let τ be left invariant and $G \setminus K \in \tau$ for each $K \in [G]^{<\omega}$. By [2, Proposition 2.7], for every n-partition $\mathcal{P}$ of G, there exists $A \in \mathcal{P}$ and $F \in [G]^{<\omega}$ such that $F\Delta_\tau(A) \in \tau$ and $|F| \leq n!$ This statement and above observations imply that, for any $U \in \tau$ and n-partition $\mathcal{P}$ of U, there exist $F \in [G]^{<\omega}$ and $A \in \mathcal{P}$ such that $|F| \leq n!$ and $F\Delta_\tau(A) \in \tau$. Moreover, for any pregiven $V \in \tau$, F can be chosen from V^{-1}. Indeed, we take $x \in \bigcap_{g \in F} gV$ so $F^{-1}x \subseteq V$ and $x^{-1}F\Delta_\tau(A) \in \tau$.

Question 3.1. *Let τ be a filter of neighborhoods of the identity for some group topology on a group G and let $U \in \tau$. Given any n-partition $\mathcal{P}$ of U and $V \in \tau$, do there exist $A \in \mathcal{P}$ and $F \subseteq V$ such that $FAA^{-1} \in \tau$ and $|F| \leq n!$*

By Theorem 3.2, the answer to Question 3.1 is positive with 2^{2^n} in place of $n!$, .

Question 3.2. *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for any group G, a filter τ of a group topology on G, $U \in \tau$ and an n-partition $\mathcal{P}$ of U, there are $A \in \mathcal{P}$ and $K \in [G]^{<\omega}$ such that $K\Delta_\tau(A) \in \tau$ and $|K| \leq f(n)$? If yes, then can K be chosen from pregiven $V \in \tau$?*

We conjecture the positive answer to Question 3.2 with $f(n) = 2^{2^n}$ (or even with $f(n) = n!$).

References

[1] T. Banakh, I. Protasov, S. Slobodianiuk, *Syndedic submeasures and partitions of G-spaces and groups*, Intern. J. Algebra Comp. **23** (2013), 1611–1623.

[2] T. Banakh, I. Protasov, S. Slobodianiuk, *Densities, submeasures and partitions of G-spaces and groups*, Algebra Discrete Math. **17** (2014), N2, 193–221, preprint (http://arxiv.org./abs/1303.4612).

[3] T. Banakh, I. Protasov, S. Slobodianiuk, *Scattered subsets of groups*, Ukr. Math. J. **65** (2015), 304–312, preprint (http://arxiv.org/abs/1312.6946).

[4] A. Bella, V. Malykhin, *On certain subsets of a group*, Questions, Answers Gen Topology **17** (1999), no.2, 183–197.

[5] A. Bella, V. Malykhin, *On certain subsets of a group II*, Questions, Answers Gen Topology **19** (2001), no.1, 81–94.

[6] V. Bergelson, N. Hindman, R. McCutcheon, *Notes of size and combinatorial properties of quotient sets in semigroups*, Topology Proceedings **23** (1998), 23–60.
[7] J. Erde, *A note on combinatorial derivation*, preprint (http:/arxiv.org/abs/1210.7622).
[8] N. Hindman, D. Strauss, Algebra in the Stone-Čech compactification, 2nd edition, de Gruyter, 2012.
[9] Ie. Lutsenko, I. Protasov, *Sparse, thin and other subsets of groups*, Intern. J. Algebra Comp. **19** (2009), 491–510.
[10] Ie. Lutsenko, I. Protasov, *Relatively thin and sparse subsets of groups*, Ukr. Math. J. **63** (2011), 216–225.
[11] V. Malykhin, I. Protasov, *Maximal resolvability of bounded groups*, Topology Appl. **20** (1996), 1–6.
[12] V.D. Mazurov, E.I. Khukhro (eds), *Unsolved problems in group theory, the Kourovka notebook*, 13-th augmented edition, Novosibirsk, 1995.
[13] I.V. Protasov, *Ultrafilters and topologies on groups*, Siberian Math. J. **34** (1993), 163–180.
[14] I.V. Protasov, *Selective survey on Subset Combinatorics of Groups*, Ukr. Math. Bull. **7** (2011), 220–257.
[15] I.V. Protasov, *The combinatorial derivation*, Appl. Gen. Topology **14** (2013), N2, 171–178.
[16] I.V. Protasov, T.Banakh, Ball Structures and Colorings of Groups and Graphs, Math. Stud. Monogr. Ser., Vol. 11, VNTL, Lviv, 2003.
[17] I. Protasov, S. Slobodianiuk, *Prethick subsets in partitions of groups*, Algebra Discrete Math. **14** (2012), 267–275.
[18] I. Protasov, S. Slobodianiuk, *Ultracompanions of subsets of groups*, Comment. Math. Univ. Corolin. **55** (2014), N2, 257–265.
[19] I. Protasov, S. Slobodianiuk, *Partitions of groups*, Matem. Stud. **42** (2014), 115–128.
[20] I.V. Protasov, M. Zarichnyi, General Asymptology, Math Stud. Monogr. Ser., Vol. 12, VNTL, Lviv 2007.

Department of Cybernetics, Kyiv University, Volodymyrska 64, 01033, Kyiv, Ukraine
E-mail address: i.v.protasov@gmail.com;

Department of Mechanics and Mathematics, Kyiv University, Volodymyrska 64, 01033, Kyiv, Ukraine
E-mail address: slobodianiuk@yandex.ru